# ME 321: Fluid Mechanics-I

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## Lecture - 08 (21/06/2025)
### Fluid Dynamics: Bernoulli Equation

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# Bernoulli Equation

## Differential Control Volume (CV) Analysis

The differential control volume chosen is fixed in space and bounded by flow streamlines, is thus an element of a **stream tube** as shown in Figure. The length of the control volume is ds.

Applying the conservation of mass and momentum equations to such a control volume results a simple differential equation describing the flow and by integrating it along a streamline will give the famous Bernoulli equation.

During the analysis, following assumptions will be considered:

(i) Steady flow
(ii) Inviscid flow (no friction, ideal fluid flow, $\mu$ = 0)
(iii) Incompressible flow (density is constant)
(iv) Irrotational flow (zero vorticity)
(v) Flow along a streamline

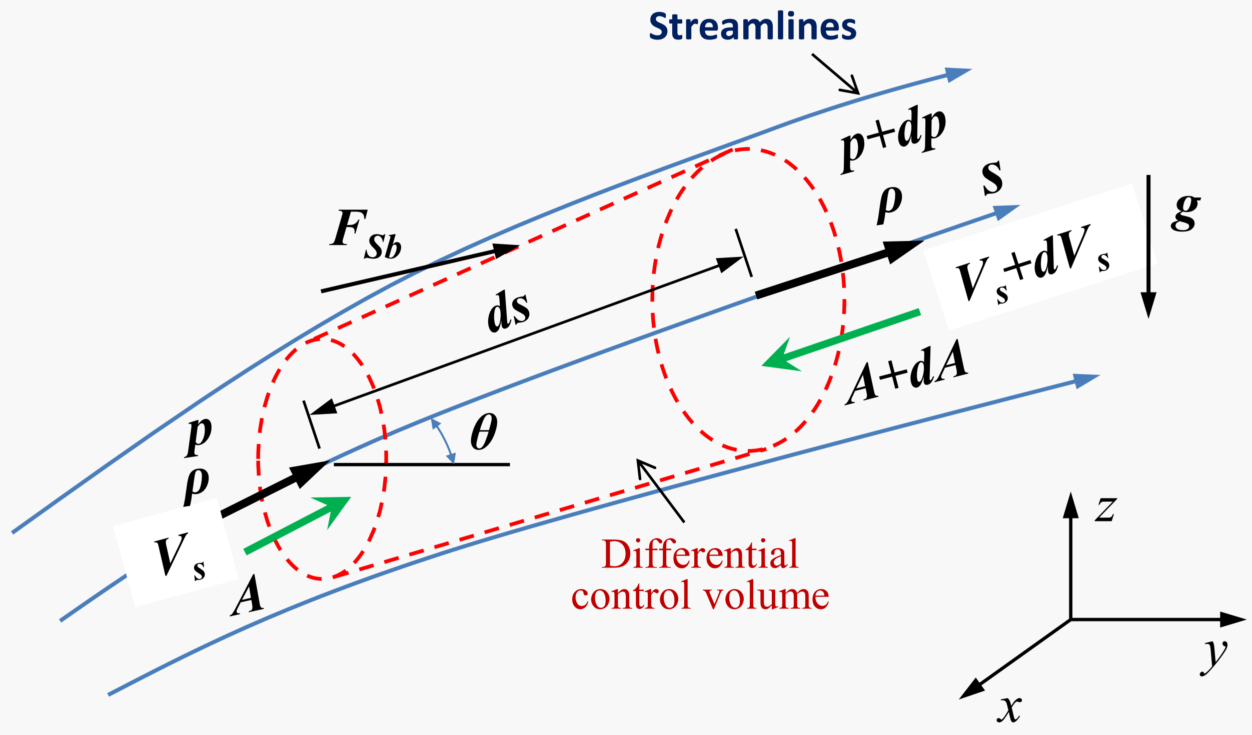



**Fig.** Differential control volume for momentum analysis of flow through a **stream tube**

Because the control volume is bounded by streamlines, the flow across the bounding surfaces occurs only at the end sections. These are located at coordinates, s and s+ds, measured along the central streamline.

Properties at the inlet are assigned arbitrary symbolic values. Properties at the outlet section are assumed to increase by differential amounts. Thus at s+ds, the flow speed is assumed to be $V_s+dV_s$, and so on.

# Differential Control Volume Analysis

## a. Continuity equation

$$\overset{=0}{\cancel{\frac{d}{dt}}}\int_{CV}\rho\, d\forall + \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

$$\Rightarrow \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

$$\Rightarrow -\rho V_s A + \left\{\rho\left(V_s + dV_s\right)\left(A + dA\right)\right\} = 0$$

$$\Rightarrow -\rho V_s A + \rho\left(V_s + dV_s\right)\left(A + dA\right) = 0$$

$$\Rightarrow -\rho V_s A + \rho V_s A + \rho V_s dA + \rho A dV_s + \overset{=0}{\cancel{\rho dA dV_s}} = 0$$

product of two differentials $dAdV_s$ is insignificant compared to other terms.

$$\boxed{\Rightarrow V_s dA + A dV_s = 0}$$

**continuity equation** for the differential control volume

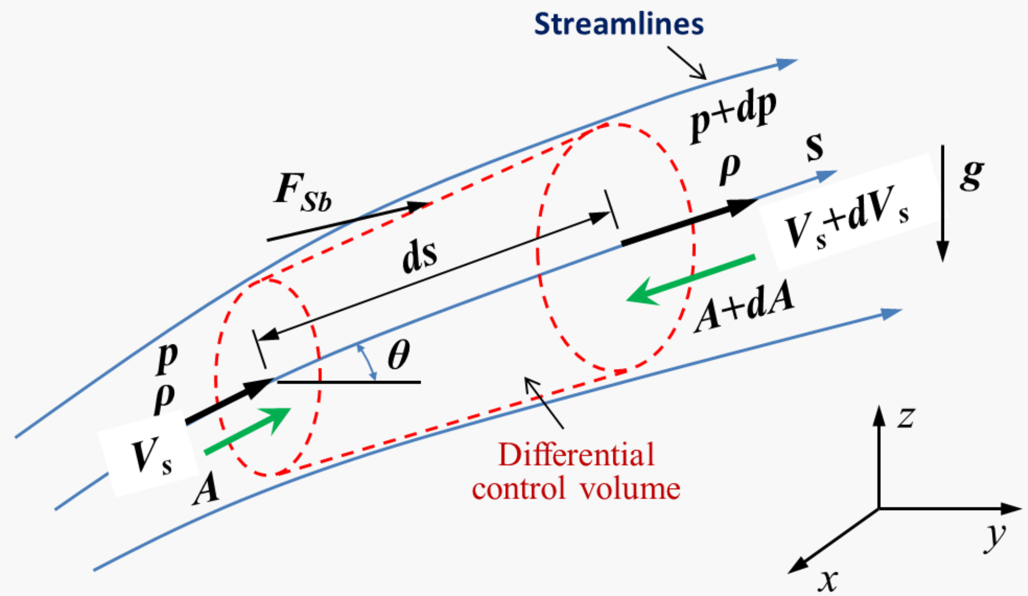



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

# Differential Control Volume Analysis

## b. Momentum equation (along steamwise direction)

$$\sum\left(\vec{F}_S+\vec{F}_B\right)=\cancelto{=0}{\frac{d}{dt}}\int_{\mathrm{CV}}\vec{\mathbf{V}}\rho\, d\forall+\int_{\mathrm{CS}}\vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \sum\left(F_{S_s}+F_{B_s}\right)=\int_{\mathrm{CS}}V_s\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

(along streamwise direction, $s$)

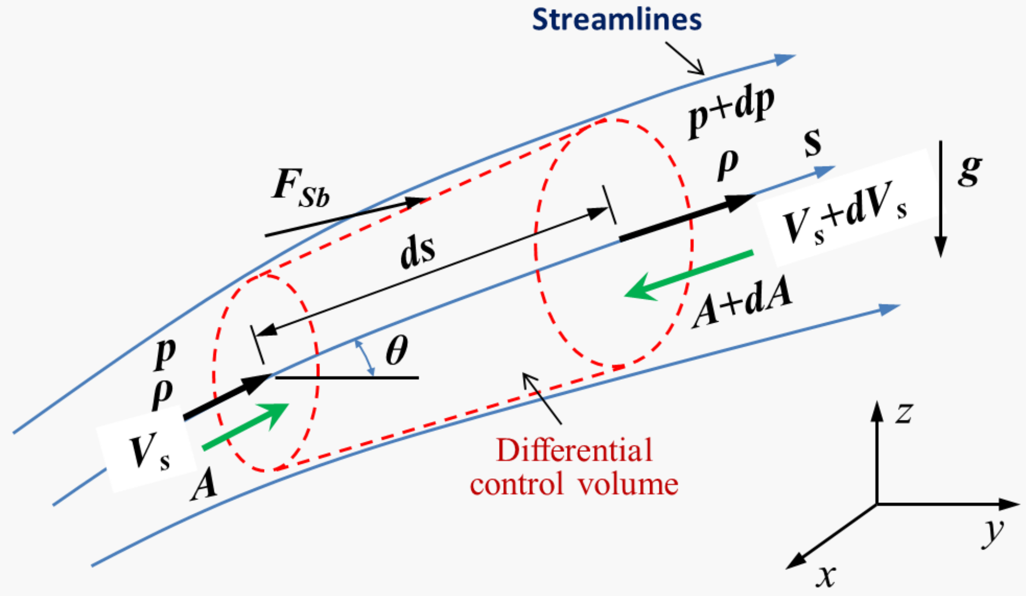



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

**Surface force** only comes from the pressure ($\mu$ = 0):

$$F_{S_s}=pA-(p+dp)(A+dA)+\left(p+\frac{dp}{2}\right)dA$$

(Left surface) (Right surface) (bounding stream surface)

(average pressure acting on the bounding surface ($p+\frac{1}{2}dp$ ) times the area component of the stream surface in $s$-direction, dA)

$$\Rightarrow F_{S_s}=pA-pA-pdA-Adp-\cancelto{=0}{dpdA}+pdA+\cancelto{=0}{\frac{dp}{2}dA}$$

product of two differentials $dpdA$ is insignificant compared to other terms.

$$\boxed{\Rightarrow F_{S_s}=-Adp}$$

# Differential Control Volume Analysis

**Body force** acting along *s*-direction:

$$F_{B_s} = \rho(-g\sin\theta)\left(A+\frac{dA}{2}\right)ds$$

$$\Rightarrow F_{B_s} = -\rho g\left(A+\frac{dA}{2}\right)dz$$

$$\Rightarrow F_{B_s} = -\rho gAdz - \rho g\,\underbrace{\frac{dA}{\cancel{2}}dz}_{=0}$$

$$\boxed{\Rightarrow F_{B_s} = -\rho gAdz}$$

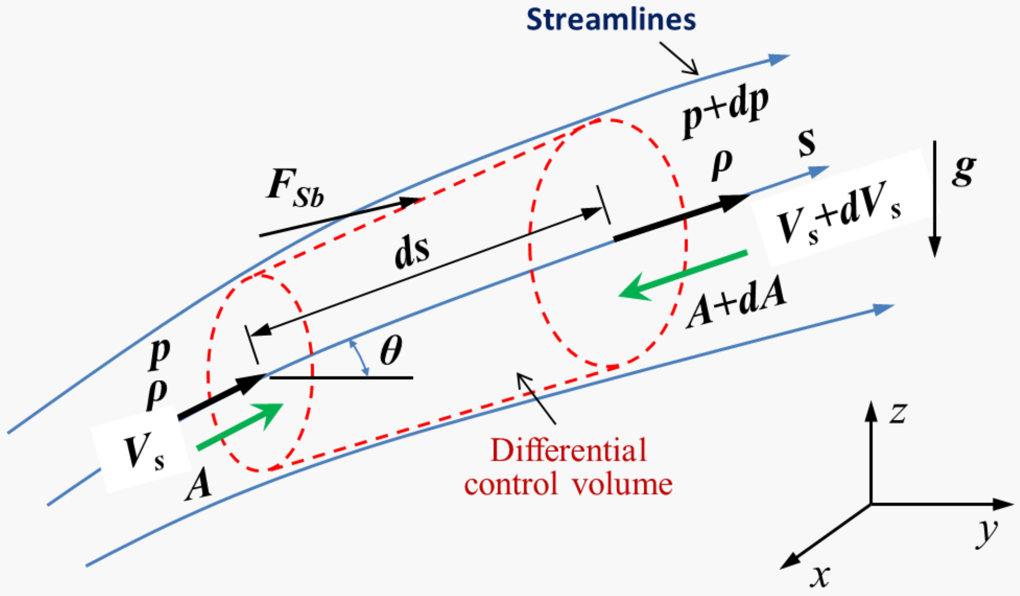



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

$$\boxed{\sin\theta = \frac{dz}{ds}}$$

# Differential Control Volume Analysis

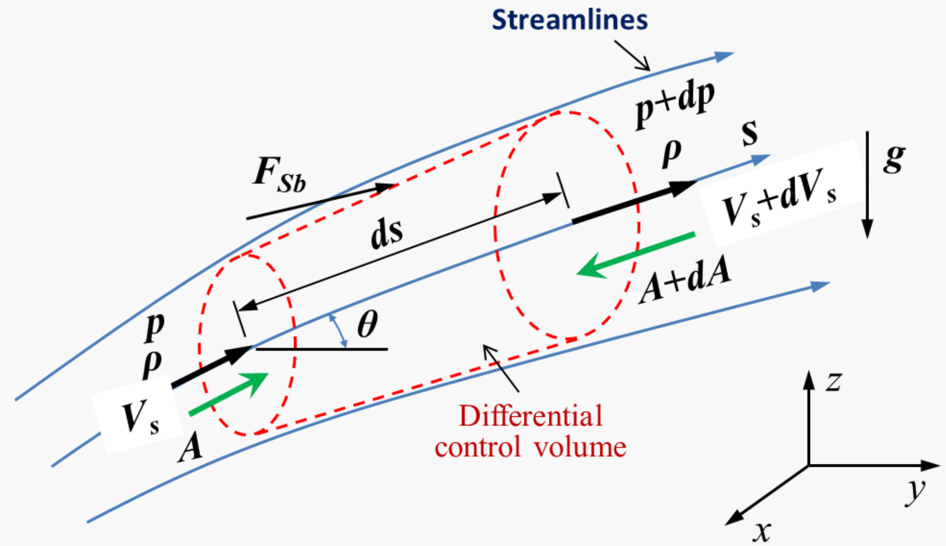



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

Right hand side of momentum equation:

$$\int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = V_s\left(-\rho V_s A\right) + \left(V_s + dV_s\right)\left\{\rho\left(V_s + dV_s\right)\left(A + dA\right)\right\}$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2}A + \left(V_s + dV_s\right)\left(\rho V_s A + \underbrace{\rho V_s dA + \rho A dV_s}_{=0} + \overset{=0}{\cancel{\rho dV_s dA}}\right)$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2}A + \left(V_s + dV_s\right)\left(\rho V_s A\right)$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2}A + \rho V_s^{\,2}A + \rho V_s A dV_s$$

$$V_s dA + A dV_s = 0 \text{ (continuity )}$$
$$\therefore \rho V_s dA + \rho A dV_s = 0$$

$$\boxed{\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = \rho V_s A dV_s}$$

# Differential Control Volume Analysis

**Now, momentum equation** **(along steamwise direction)**

$$\sum\left(F_{S_s}+F_{B_s}\right)=\int_{CS}V_s\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow -Adp-\rho gAdz=\rho V_s A dV_s$$

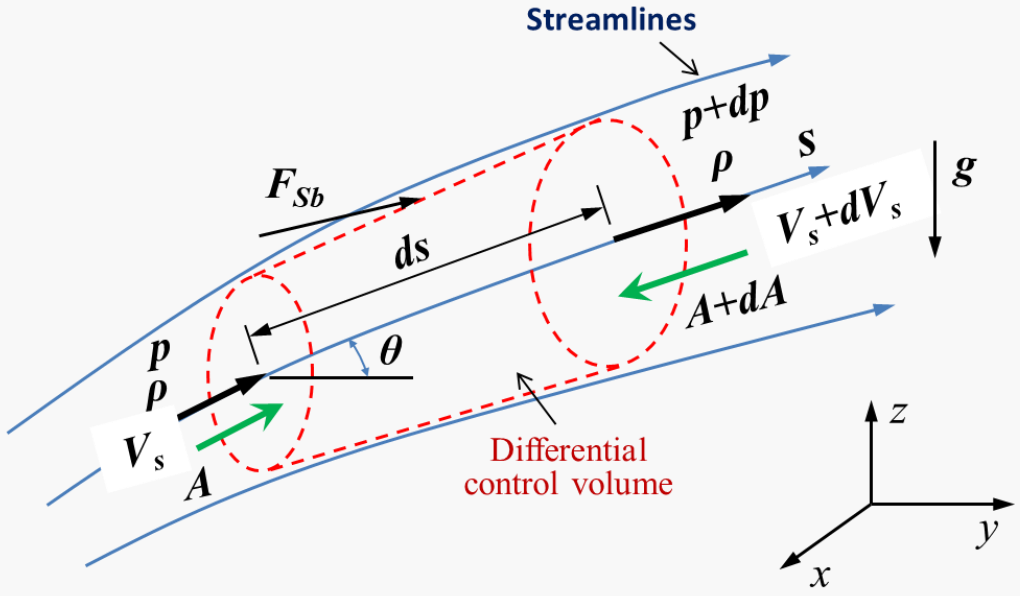



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

**Dividing both sides by $\rho A$:**

$$\Rightarrow -\frac{dp}{\rho}-gdz=V_s dV_s=d\left(\frac{V_s^{\,2}}{2}\right)$$

$$\boxed{\Rightarrow \frac{dp}{\rho}+d\left(\frac{V_s^{\,2}}{2}\right)+gdz=0}$$

**Euler differential equation**

**for steady inviscid incompressible fluid flow .**

# Differential Control Volume Analysis

Integrate the Euler equation along a streamline:

$$\int \frac{dp}{\rho} + \int d\left(\frac{V_s^{\,2}}{2}\right) + \int g dz = \text{Constant}$$

$$\boxed{\Rightarrow \frac{p}{\rho} + \frac{V_s^{\,2}}{2} + gz = \text{Constant}}$$

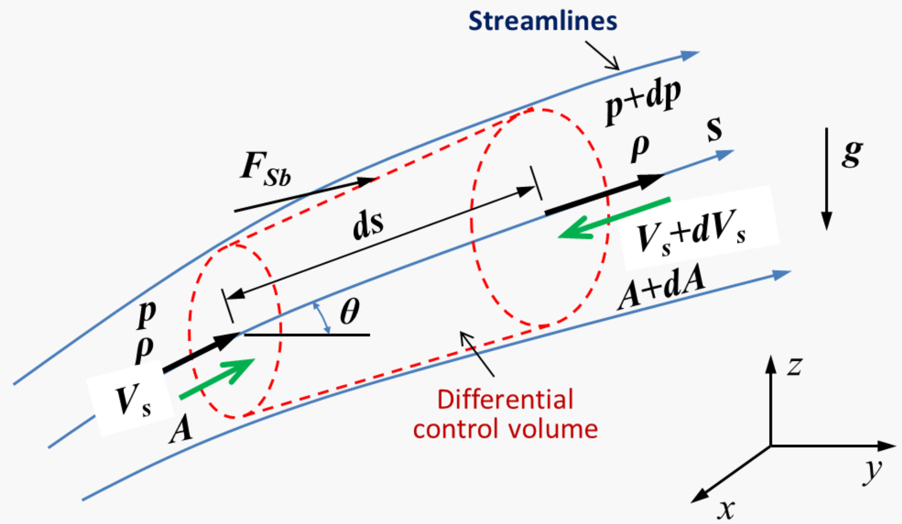



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

Suffix *s* can be dropped conveniently (since fluid flows along the streamline)

$$\boxed{\Rightarrow \frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{Constant}}$$

**Bernoulli equation.**

(Most famous and mostly used equation in fluid dynamics)

# Differential Control Volume Analysis

The Bernoulli equation is a momentum-based force relation.
It may be interpreted as an idealized energy relation.

$$\underbrace{\frac{p}{\gamma}}_{\text{Pressure head}} + \underbrace{\frac{V^2}{2g}}_{\text{Velocity head}} + \underbrace{z}_{\text{Potential head}} = \text{Constant}$$

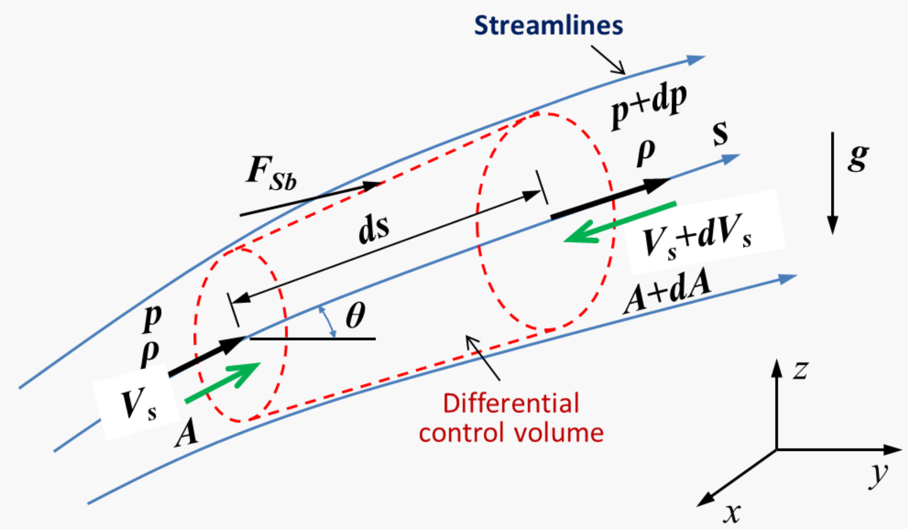



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

Subjected to the following restrictions in fluid flow:

(i) Steady flow
(ii) Inviscid flow (no friction, ideal fluid flow, $\mu = 0$)
(iii) Incompressible flow (density is constant)
(iv) Irrotational flow
(v) Flow along a streamline

Although no real flow satisfies all these restrictions (especially the second one), we can approximate the behavior of many flows.

# Differential Control Volume Analysis

Considering any two points along a streamline, **Bernoulli Equation** yields:

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^{\,2}}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^{\,2}}{2g} + z_2}$$

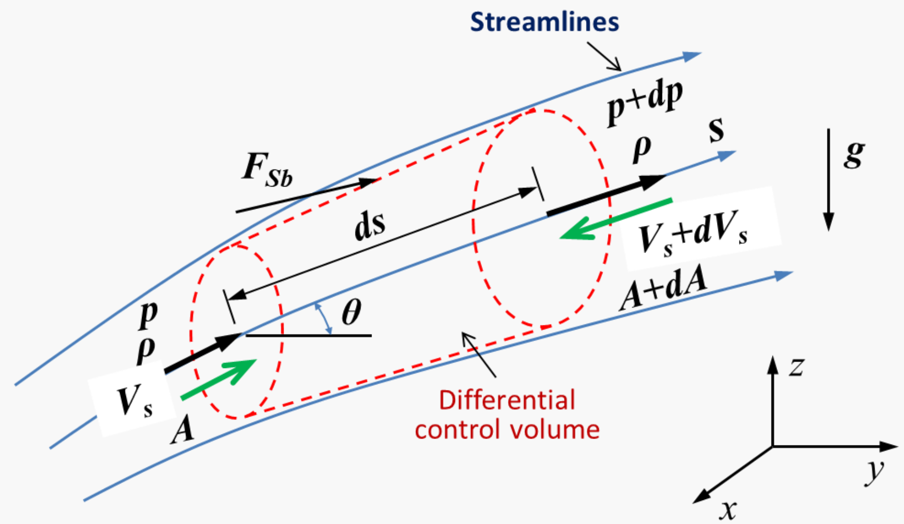



**Fig.** Differential control volume for momentum analysis of flow through a stream tube

# Validity of Bernoulli equation

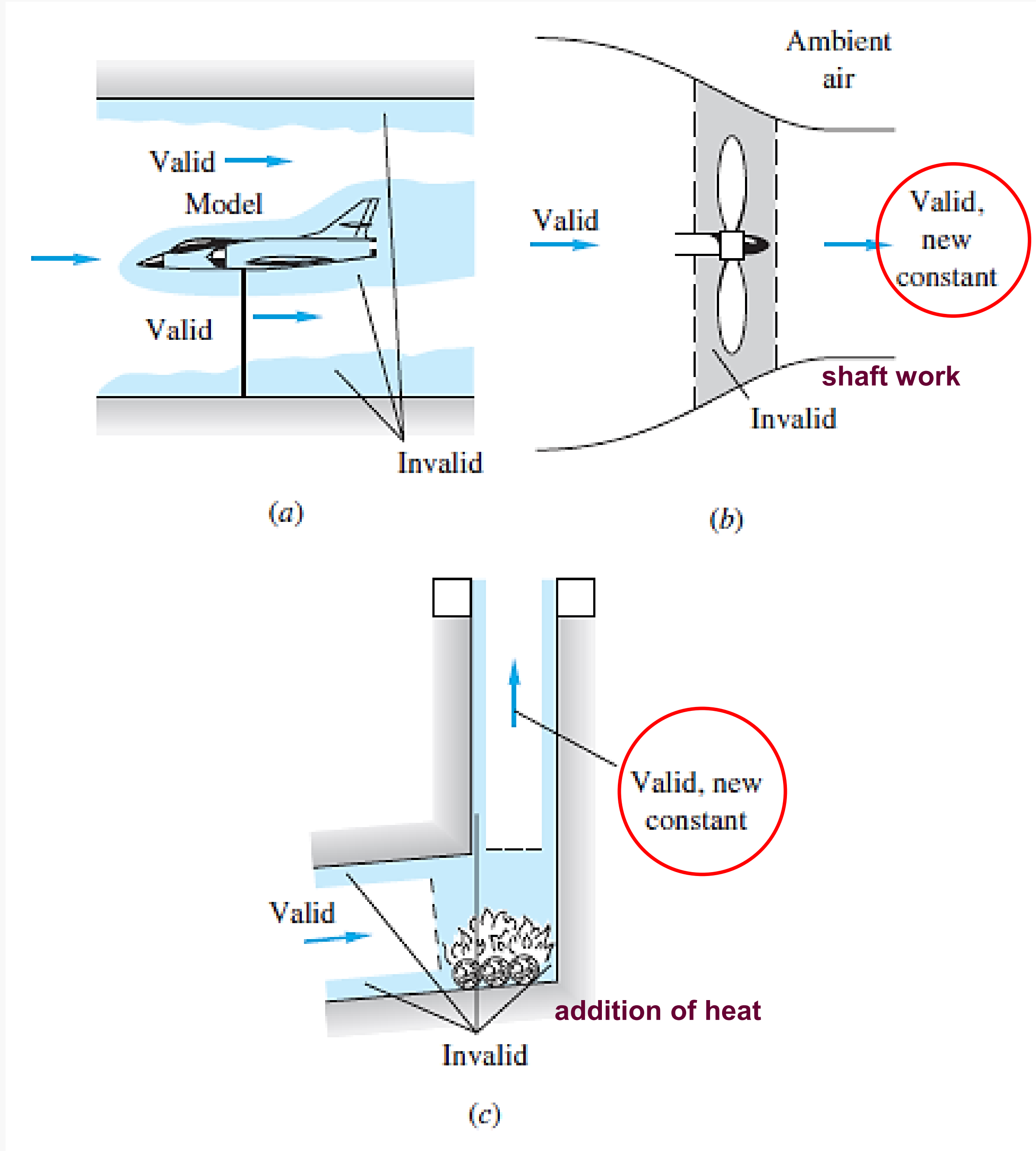



**Fig. 3.13** Illustration of regions of validity and invalidity of the Bernoulli equation: (*a*) tunnel model, (*b*) propeller, (*c*) chimney.

# Static, Dynamic & Stagnation Pressure

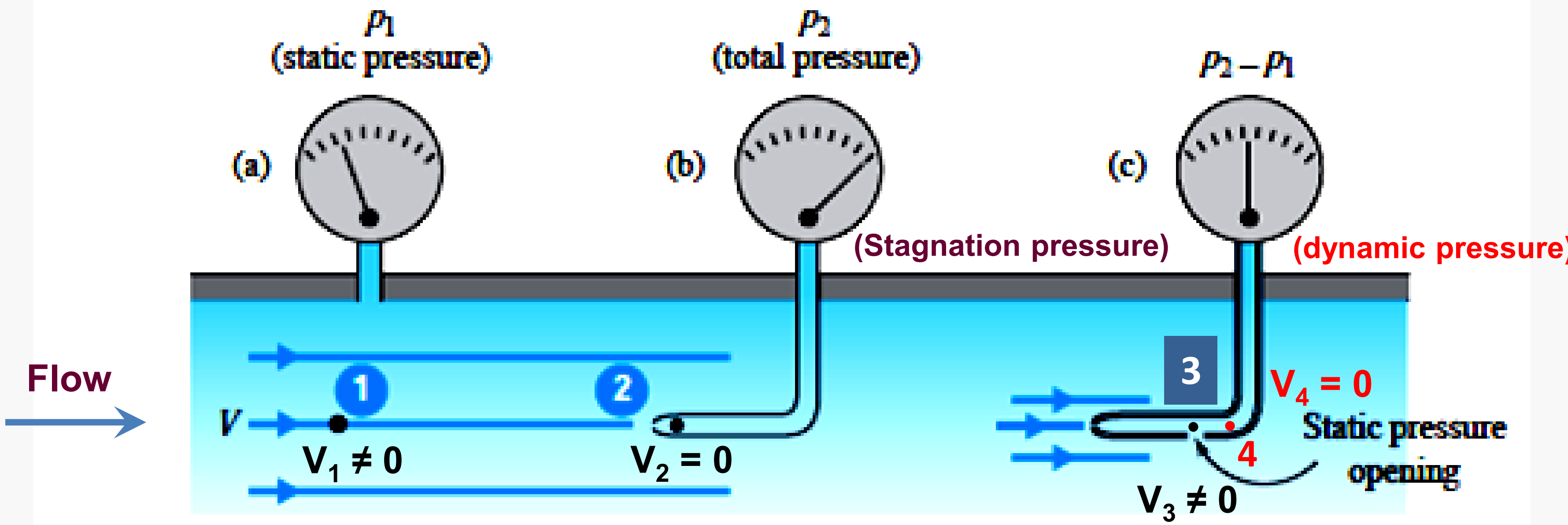



Fig. 3.18 Pressure probes: (a) piezometer; (b) pitot probe; (c) pitot-static probe.

$$\frac{1}{2}\rho V^2$$

$$\frac{p_1}{\gamma}+\frac{V_1^{\ 2}}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^{\ 2}}{2g}+z_2$$

$$\Rightarrow \frac{p_1}{\gamma}+\frac{V_1^{\ 2}}{2g}=\frac{p_0}{\gamma}+\frac{0^2}{2g}$$

dynamic pressure

$$\boxed{\Rightarrow p_0 = p_1 + \frac{1}{2}\rho V_1^{\ 2}}$$

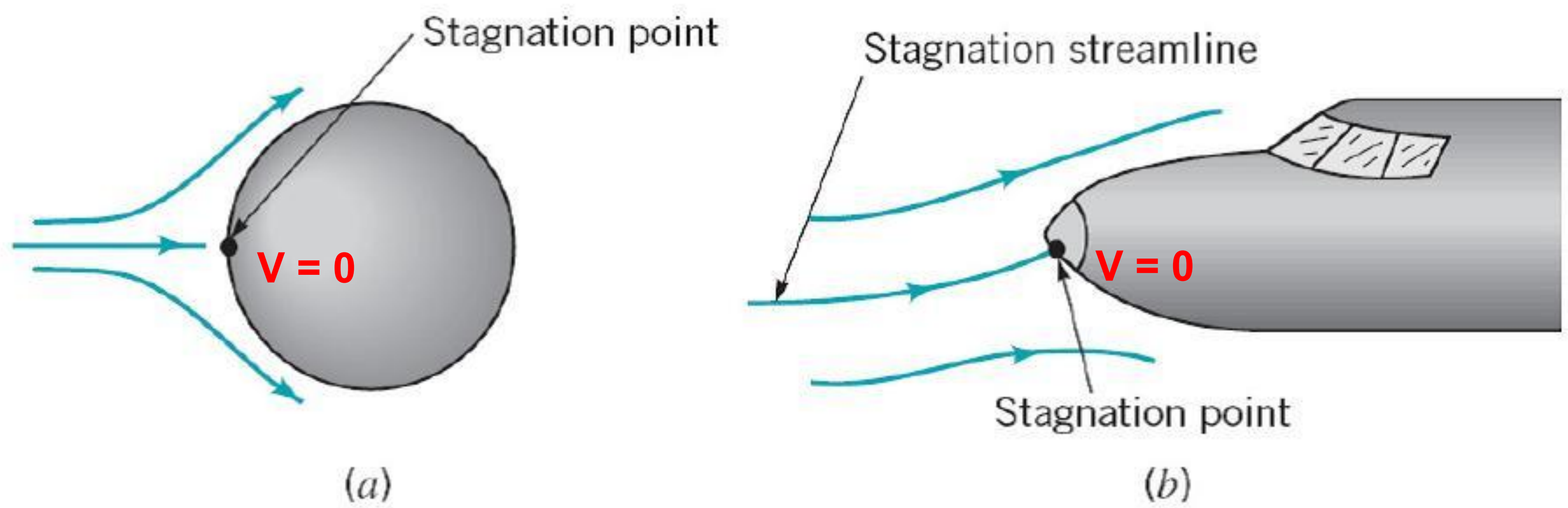
Stagnation/ total pressure

Static pressure

(Incompressible flow)

**Stagnation Pressure = Static pressure + Dynamic Pressure**

# Stagnation Point




Stagnation points on bodies in flowing fluids.

# HGL & EGL

(frictionless flow, no shaft work or heat transfer; EGL = const.)

**EGL = Energy Grade Line**

$$\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z = h_0\right)$$

**HGL = Hydraulic Grade Line**

$$\left(\frac{p}{\gamma} + z\right)$$

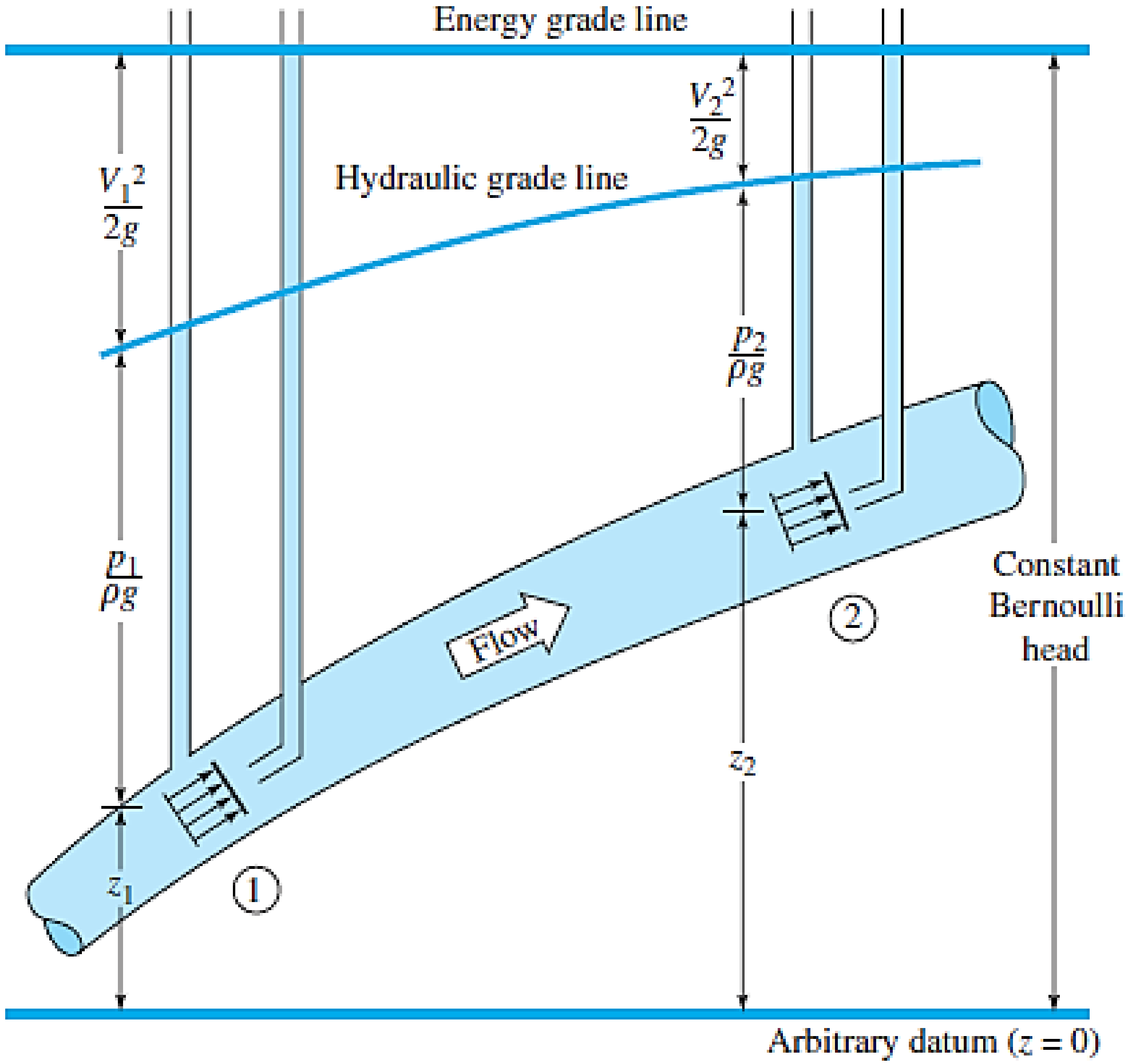



**Fig. 3.14** Hydraulic and energy grade lines for frictionless flow in a duct.